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If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a lin. transf then

$\ker(T)$ is a subspace of $\mathbb{R}^m$

$\mathrm{im}(T)$ " " " " $\mathbb{R}^n$

Last time we saw how to find a basis of $\ker(T)$, via an example.

The main idea was to use what we know about solving the lin. system

$$A X = \vec{0}$$

via RREF(A).

Ex

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$

The observation that made everything work is

$$\boxed{\ker(A) = \ker(\mathrm{RREF}(A))}$$

So we found RREF(A), and we saw that a basis for $\ker(T)$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- How about $\mathrm{im}(T)$ = col space of A ?

We have

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$

and

$$B = RREF(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: it's <u>NOT</u> true that column space of A = col. space of B (analogously to $\ker(A)$). But B is still useful.

We start with the 5 columns of A, which certainly span the column space.

Idea: get rid of redundant columns in A.

How do we use B to do this?

<u>Trick</u>: the columns in B that have leading 1's correspond to the columns of A that form our basis. (The rest are redundant.)

So in our example, a basis for $\mathrm{im}(T)$ is

$$\begin{bmatrix} 1 \\ -1 \\ 4 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -1 \\ 5 \\ 1 \end{bmatrix}$$

Why does this work? See part (b) of Example 1 in §3.3

E.g. Say we have a linear dependence of the columns

$$-3\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+4\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+\begin{bmatrix}3\\-4\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Corresp to

$$-3\begin{bmatrix}1\\-1\\4\\3\end{bmatrix}+4\begin{bmatrix}2\\-1\\5\\1\end{bmatrix}+\begin{bmatrix}-5\\1\\-8\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Row operations change the vectors but not the dependence.

Corollary (Th 3.3.6) $\dim(\operatorname{im}(A)) = \dim(\text{col. space of } A)$
$= \#$ of leading 1's in RREF(A)
$= \operatorname{rank}(A)$

Theorem (3.3.7) (Rank-Nullity Theorem)

For any $n\times m$ matrix A we have $A:\mathbb{R}^m \to \mathbb{R}^n$ and

$$\dim(\ker(A)) + \dim(\operatorname{im}(A)) = m$$

called the nullity of A $\qquad = \operatorname{rank}(A)$

Now put a few things together.

- Start with $\mathbb{R}^n$.
- $\vec{e}_1, \dots, \vec{e}_n$ clearly form a basis (e.g. ...)
- $\Rightarrow$ any basis of $\mathbb{R}^n$ has n elements (Th. 3.3.2)
- Now take a set of n vectors $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$.

We want conditions that tell us if these vectors form a basis.

Form the matrix $A = \begin{bmatrix} \vec{v}_1 & | & \dots & | & \vec{v}_n \end{bmatrix}$ $(n \times n)$.

Th 3.2.10 says $\boxed{\vec{v}_1, \dots, \vec{v}_n \text{ form a basis}}$ iff every vector $\vec{b} \in \mathbb{R}^n$ can be written uniquely as a lin comb of $\vec{v}_1, \dots, \vec{v}_n$.

This means $A X = \vec{b} \Leftrightarrow \begin{bmatrix} \vec{v}_1 & | & \dots & | & \vec{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$ has a unique solution no matter what $\vec{b}$ is.

We saw this holds iff $RREF(A) = I_n$

This is true iff $rk(A) = n$

$\boxed{\text{iff } A \text{ is invertible.}}$

This is the content of Th 3.3.9

Th 3.3.10 gives 9 equiv conditions for a matrix to be invertible.

Next section:

§ 3.4 Coordinates

3.4 Coordinates

Coordinates are one of the "great ideas" of mathematics. René Descartes (1596–1650) is credited with having introduced them, in an appendix to his treatise *Discours de la Méthode* (Leyden, 1637). Myth has it that the idea came to him as he was laying on his back in bed one lazy Sunday morning, watching a fly on the ceiling above him. It occurred to him that he could describe the position of the fly by giving its distance from two walls.